Periodic patterns with conservation laws: the Whitham equations and rigorous long-time asymptotics

Kevin Zumbrun

Collaborators: Blake Barker, Mathew Johnson, Pascal Noble, Miguel Rodrigues

Department of Mathematics
Indiana University

Sponsored by NSF Grants no. DMS-0300487 and DMS-0801745

October 3, 2013
Periodic patterns and traveling waves arise in optics, biology, chemistry, etc. Their analysis is difficult due to *purely essential spectra, lack of spectral gap*. Two triumphs are formal understanding by modulation expansion, etc., and rigorous treatment by renormalization techniques [Schneider95, ...]

However, systems with a conservation law—arising commonly in hydrodynamic settings—don’t fit this theory.

**Goals of this talk:**
- Exposition of existing and new results in common framework.
- Link to hyperbolic-parabolic conservation laws (my background) through formal modulation equations.
Consider a periodic traveling wave solution

\[ u(x, t) = \bar{u}(x - ct), \quad \bar{u}(X) = \bar{u}(0), \]

of a parabolic system of balance laws

\[ u_t + f(u)_x + g(u) = u_{xx}, \quad u \in \mathbb{R}^n, \quad g = \begin{pmatrix} 0 \\ g \end{pmatrix} \]

(equivalently stationary solution of \( u_t - cu_x + f(u)_x + g(u) = u_{xx} \)).

**Extremes:** \( g = 0 \) (conservation law); \( f = 0 \) (reaction diffusion).
(related) Example models

St. Venant equations for shallow water flow on an incline

\[ \nu_t - u_x = 0 \]
\[ u_t + p(v)_x = \left( \frac{u_x}{\nu^2} \right)_x + 1 - \nu u^2. \]

Generalized Kuramoto–Sivashinsky equation (gKS)

\[ u_t + (u^2/2)_x + \delta u_{xx} + \varepsilon u_{xxx} + \delta u_{xxxx} = 0. \]

One-dimensional viscoelasticity with surface energy

\[ \nu_t - u_x = \nu_{xx}, \]
\[ u_t + \sigma(v)_x = u_{xx} \]
Other models

Complex Ginzburg Landau equation (cGL)

\[ A_t = (1 + i\alpha)A_{xx} + \mu A - (1 + i\beta)A|A|^2 + (\gamma_1 + i\gamma_2)A|A|^4, \quad A \in \mathbb{C} \]

Matthews–Cross model

\[ A_t = (1 + i\alpha)A_{xx} + \mu A - (1 + i\beta)A|A|^2 + (\gamma_1 + i\gamma_2)A|A|^4, \]
\[ B_t = \sigma B_{xx} + \mu (|A|^2)_{xx}, \quad A \in \mathbb{C}, \quad B \in \mathbb{R} \]

gKdV: rather, dispersive/Hamiltonian. + hidden conservation law

\[ u_t^2 + \ldots \text{ (modulational) stability an open question.} \]
\[ u_t + f(u)_x = u_{xxx} \]
Existence: the traveling wave ODE

(Case $g \equiv 0$) Substituting $u = \bar{u}(k(x - ct))$ into $u_t + f(u)_x = u_{xx}$ (hence fixing period one) and integrating in $x$ yields ODE

$$k \bar{u}' = f(\bar{u}) - c\bar{u} - q, \quad (u_0, q, s, X) \equiv \text{constant}. \quad (1)$$

(H1) $f \in C^5$,
(H2) The set of periodic traveling waves near $\bar{u}$ form a smooth $(n + 2)$-dimensional manifold

$$\{\bar{u}^M,k(kx + \omega(M, k)t - \alpha)\}, \quad (2)$$

with $k = 1/X$ = wave number, $M =$mean over one period of $\bar{u}$, $\alpha \in R$ translation. ($(M, k)$ parametrization convenient later.)

(compare reaction diffusion case: just $k \in R^1$; think Hopf bifurcation.)
Stability: Bloch decomposition

Linearized equations:

\[ v_t = Lv := (\partial_x^2 - \partial_x A)v, \quad A := Df(\bar{u}) \text{ periodic.} \]

Bloch decomposition, \( u \in L^2(\mathbb{R}) \):

\[ u(x) = \left( \frac{1}{2\pi} \right) \int_{-\pi}^{\pi} e^{i\xi x} \check{u}(\xi, x) d\xi, \]

\[ \check{u}(\xi, \cdot) := \sum_{j \in \mathbb{Z}} \hat{u}(\xi + 2\pi j) e^{2\pi ij \cdot} \in L^2[0, 1]_{\text{periodic}}. \]

Inverse Bloch transform representation:

\[ e^{Lt} u_0 = \left( \frac{1}{2\pi} \right) \int_{-\pi}^{\pi} e^{i\xi x} e^{L \xi t} \check{u}_0(\xi, x) d\xi, \quad (3) \]

\[ L_\xi := e^{-i\xi x} Le^{i\xi x} = (\partial_x + i\xi)^2 - (\partial_x + i\xi)A. \]
(D1) $\sigma(L_\xi) \subset \{ \text{Re}\lambda < 0 \}$ for $\xi \neq 0$.

(D2) $\text{Re}\sigma(L_\xi) \leq -\theta|\xi|^2$, $\theta > 0$, for $\xi \in \mathbb{R}$ and $|\xi|$ sufficiently small.

(D3) $\lambda = 0$ is an eigenvalue of $L_0$ of (minimum) multiplicity $n + 1$.

(H1)-(H2) and (D1)–(D3) $\Rightarrow$ there exist $n + 1$ smooth eigenvalues

$$\lambda_j(\xi) = -ia_j\xi + o(|\xi|)$$

of $L_\xi$ bifurcating from $\lambda = 0$ at $\xi = 0$, $a_j$ constant.

(H3) The coefficients $a_j$ in (4) are distinct.

(RD case: $\lambda = 0$ simple. Here, generically Jordan block.)
First Main Result: nonlinear stability

**Theorem (Johnson-Z10)**

Let $\tilde{u}$ be a solution of $u_t + f(u)_x = u_{xx}$. Assuming (H1)–(H3), (D1)–(D3), and $E_0 := \|\tilde{u} - \bar{u}\|_{L^1 \cap H^4}|_{t=0}$ sufficiently small,

$$\|\tilde{u} - \bar{u}(\cdot - \psi)\|_{L^p(t)} \leq C(1 + t)^{-\frac{1}{2}(1-1/p)}E_0,$$

$$\|(\psi_t, \psi_x)\|_{W^{5,p}} \leq C(1 + t)^{-\frac{1}{2}(1-1/p)}E_0,$$

$$\|\tilde{u} - \bar{u}\|_{L^\infty(t)} \leq CE_0,$$

$$\|\psi(t)\|_{L^\infty} \leq CE_0$$

for some $\psi(x, t)$, $C > 0$, and all $t \geq 0$, $p \geq 2$. 

Zumbrun
Periodic patterns with conservation laws
Resolves question posed in [Oh-Z01], generalizing [Schneider95].

- Very weak decay (compare Gaussian decay of reaction diffusion case) requires more detail to prove.
- Exact analogy to viscous shock case [Mascia-Z]: \( \psi = \psi(t) \).
- Corresponding result holds for multi-d [Johnson-Z10], St-Venant [JZ-Noble10], (gKS) [Barker-J-Rodrigues-NZ11].

**Remark.** (gKS) open since 1976.
II. Whitham (averaged) equations and modulation

**Linear modulation** (review):

The Schrödinger equation $iu_t = u_{xx}$ admits oscillatory modes $A e^{i(kx + \omega(k)t)}$, where $k = 1/X$ is wavenumber and $\omega$ temporal frequency, with dispersion relation $\omega(k) = -k^2$ (nonlinear).

*Phase velocity* (speed of single wave) is thus $c_p = \omega(k)/k = -k$.

Superposition $\int_{\mathbb{R}} \alpha(k) e^{i(kx + \omega(k)t)} \, dk$ of frequencies $k \sim k_*$ gives a *modulated wave packet* $\sim A(x + \omega'(k_*) t) e^{i(k_* x + \omega(k_*) t)}$

$$A(z) = \int_{\mathbb{R}} \alpha(k_* + h) e^{ihz} \, dh, \quad h = k - k_*,$$

or *group velocity* $c_g = \omega'(k_*)$.

*(particle/wave duality, electromagnetic transmission, etc.)*
Nonlinear modulation the Whitham equations

(Case \( g = 0 \)) Rescaling \((x, t) \rightarrow (\epsilon x, \epsilon t)\): \( u_t + f(u)_x = \epsilon u_{xx} \), and carrying out WKB expansion

\[
  u^\epsilon(x, t) = u^0 \left( x, t, \frac{\psi(x, t)}{\epsilon} \right) + \epsilon u^1 \left( x, t, \frac{\psi(x, t)}{\epsilon} \right) + \cdots , \tag{5}
\]

as \( \epsilon \to 0 \), matching terms, yields at \( \epsilon^{-1} \) order

\[
  u^0(x, t, \cdot) = \bar{u}^M(x, t), k(x, t) \quad \text{(recall parametrization (2))}, \quad \text{and at } \epsilon^0 \text{ order}
\]

\[
  M_t + F_x = 0, \\
  k_t + \omega_x = 0, \tag{6}
\]

where \( M, F \) are averages of \( u^0(x, t), f(u^0(x, t)) \), and \( k = \psi_x \), \( \omega = ck = -\psi_t \), and \( c = -\psi_t/\psi_x \) are wave number, frequency, and wavespeed, with nonlinear dispersion relation \( \omega = \omega(k, M) \).

For RD case \((g \text{ full rank, } M = \emptyset)\), reduces to scalar conservation law, characteristic=group velocity \( c = \omega'(k) \).
Remarks

• For nontrivial $M$, it is a system of conservation laws, richer behavior (e.g., “viscoelastic behavior” of KS cells [Frisch76]).

• Formally, stability of a single wave $\bar{u} \sim$ stability of $(M, k) \equiv (\bar{M}, \bar{k})$, hyperbolicity of (8)

• Multiple characteristic speeds (linear group velocities) prevent renormalization methods. Our analysis closer to viscous shock stability.
Relation to spectral stability

Linearized dispersion relation

\[ \hat{\Delta}(\xi, \lambda) := \det \left( \lambda \text{Id} + i\xi \frac{\partial (F, \omega)}{\partial (\bar{M}, \bar{k})} (\bar{M}, \bar{k}) \right) = 0, \]

roots \( \lambda = \tilde{a}_j \xi, \ j = 1, \ldots, n + 1. \)

Lemma (Frisch76, Serre2005, Oh-Z06, Noble-Rodrigues11)

\( a_j = \tilde{a}_j, \) where \( a_j \) as in (4) is the coefficient in spectral expansion \( \lambda_j(\xi) = -i\xi a_j + \ldots. \)

Corollary (Rigorous justification)

Hyperbolicity of (8) is necessary for stability of \( \bar{u}. \)

- \( a_j \) distinct \( \Leftrightarrow \) strict hyperbolicity.
- Likewise for \( gKdV \) [Serre2005, Johnson-Z09]. (New WKB.)
- Remarkable since Jordan block \( (\Rightarrow \) expect square root splitting).
Illustration for Kuramoto–Sivashinsky (KS)

Figure: viscoelastic behavior of perturbed KS wave
Theorem (JNRZ13)

Defining \((\mathcal{M}_W, \kappa_W)\) and \(\Psi_W\) to be solutions of the Whitham equations with appropriate initial data, we have, for \(t \geq 0\), \(2 \leq p \leq \infty\),

\[
\| \tilde{u}(\cdot, t) - U^{\mathcal{M}(\cdot, t), \kappa(\cdot, t)}(\Psi(\cdot, t)) \|_{L^p(\mathbb{R})} \lesssim E_0 \ln(2 + t) (1 + t)^{-\frac{3}{4}},
\]

\[
\| \tilde{u}(\cdot, t) - U^{\mathcal{M}_W(\cdot, t), \kappa_W(\cdot, t)}(\Psi_W(\cdot, t)) \|_{L^p(\mathbb{R})} \lesssim E_0 (1 + t)^{-\frac{1}{2}(1-1/p)+\eta}.
\]

(7)

- Faster rate than stability estimate. Includes also RD case.
- Multiple linear group velocities \(\Rightarrow\) renormalization methods don’t (seem to) apply. NEW METHODS OF ANALYSIS.
Note: “appropriate data” includes localized \((L^1 \cap H^s)\) \textit{plus} also nonlocalized change in phase \(\phi\). This corresponds in Whitham to localized data, since \(k \sim \psi_x\). Interesting linear cancellation, at first sight, analogous to \(\partial_x G \ast (\rho f) \sim G \ast \rho \partial f_x\). See [JNRZ13, ARMA].

Still more nonlinear, involving nonlocal perturbation in \(k\), are (time-periodic) \textit{defect solutions} involving a front separating two periodic waves of different wave number, analogous to \textit{shock waves} for Whitham. Here we come full cycle—remarkably, the same phase extraction strategy can yield stability in this oscillating shock regime [BeNSZ13].

\textbf{Example.} Nozaki-Bekki hole solutions \(A_{nb}(x, t) = r(x)e^{i(\varphi(x) - \omega_0 t)}\) of (cGL).
Short feature: Viscoelastic ("bouncing") behavior of KS wave. Dots mark peaks and valleys.
Appendix: WKB expansion, expanded...

Plugging into $u_t + f(u)_x = \varepsilon u_{xx}$ the approximate solution

$$u^\varepsilon(x, t) = u^0\left(x, t, \frac{\psi(x, t)}{\varepsilon}\right) + \varepsilon u^1\left(x, t, \frac{\psi(x, t)}{\varepsilon}\right) + \cdots,$$

$\varepsilon \to 0$, and matching terms, gives at yields at $\varepsilon^{-1}$ order:

$$\psi_t u^0_\theta + \psi_x f(u^0)_\theta = \psi_x^2 u_{\theta\theta},$$

where $\theta$ denotes the fast variable in $u^j(x, t, \theta)$. Recalling $k = \psi_x$, $c = -\psi_t/\psi_x$, we obtain the traveling-wave profile ODE

$$-ck(u^0)' + kf(u^0)' = k^2(u^0)'',$$

so $u^0(x, t, \cdot) = \bar{u}^{M(x, t), k(x, t)}$ is a period-1 traveling-wave profile.
Appendix: WKB expansion, continued

At $\varepsilon^0$ order, we obtain

$$u_t^0 + f(u^0)_x = (\ldots)_\theta,$$

where we have grouped a number of terms into the perfect
derivative with fast variable $\theta$ on RHS. Integrating over one period
of $u^0$ and assuming 1-periodicity of $u^0$, gives $M_t + F_x = 0$, where
$M$ is mean of $u^0 = \bar{u}^{M,k}$ and $F(M, k)$ is mean of $f(u^0)$. Thus,

$$M_t + F_x = 0,$$
$$k_t + \omega_x = 0, \quad (8)$$

where the second equation comes from equality of mixed partials
and $k = \psi_x$, $\omega = -\psi_t$. (This nice derivation due to D. Serre.)